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2000 J. Phys. A: Math. Gen. 33 1603

(http://iopscience.iop.org/0305-4470/33/8/307)

J. Phys. A: Math. Gen. **33** (2000) 1603–1629. Printed in the UK PII: S0305-4470(00)07809-4

General Hermite and Laguerre two-dimensional polynomials

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Received 13 September 1999

Abstract. General Hermite and Laguerre two-dimensional (2D) polynomials which form a (complex) three-parameter unification of the special Hermite and Laguerre 2D polynomials are defined and investigated. The general Hermite 2D polynomials are related to the two-variable Hermite polynomials but are not the same. The advantage of the newly introduced Hermite and Laguerre 2D polynomials is that they satisfy orthogonality relations in a direct way, whereas for the purpose of orthonormalization of the two-variable Hermite polynomials two different sets of such polynomials are introduced which are biorthonormal to each other. The matrix which plays a role in the new definition of Hermite and Laguerre 2D polynomials is in a considered sense the square root of the matrix which plays a role in the definition of two-variable Hermite polynomials. Two essentially different explicit representations of the Hermite and Laguerre 2D polynomials are derived where the first involves Jacobi polynomials as coefficients in superpositions of special Hermite or Laguerre 2D polynomials and the second is a superposition of products of two Hermite polynomials with decreasing indices and with coefficients related to the special Laguerre 2D polynomials. Generating functions are derived for the Hermite and Laguerre 2D polynomials.

1. Introduction

In [1, 2], we introduced and discussed special Laguerre two-dimensional (2D) functions and in [3] the corresponding Laguerre 2D polynomials. The Laguerre 2D functions are related to the Laguerre 2D polynomials in such a way that they take into account the necessary weight factor for the orthonormalization of the latter. They are eigenfunctions of the 2D harmonic oscillator. The Laguerre 2D polynomials which contain the generalized Laguerre polynomials as a substantial part are defined in a form which is very near to the form in which the usual Laguerre polynomials are mostly applied in 2D problems. The Laguerre 2D polynomials are analogous to products of two Hermite polynomials and are related to them by a transformation which corresponds in quantum optics to the transition from linear polarization to circular polarization or for a beamsplitter to the splitting of a beam into two partial beams of equal intensity. However, there are problems which need more general kinds of polynomials related to Hermite and Laguerre polynomials, for example, the transition to general elliptical polarization or general beamsplitting. In classical optics a related problem is the transformation from Hermite–Gauss and Laguerre–Gauss beams to more general types of beams in the paraxial approximation, e.g. [4–10]. For these and other purposes, in this paper we introduce and investigate a new form of general Hermite and Laguerre 2D polynomials which make a (complex) three-parameter unification of the special Hermite and Laguerre 2D polynomials corresponding to the group $SL(2, C)$ or to its subgroups in special cases.

There already exist two-variable Hermite polynomials [11–24] which make a continuous three-parameter unification of products of two Hermite polynomials with special superpositions of such products. However, these two-variable Hermite polynomials are not orthonormalized to each other and for this purpose it is necessary, apart from weight functions, to introduce a second dual kind of two-variable Hermite polynomial and the pairs of these two sets of two-variable Hermite functions then obey biorthonormality relations. However, there is another possibility to introduce a kind of Hermite or Laguerre 2D polynomials which makes this unification and satisfies additionally orthonormality relations without introduction of a different kind of polynomials. This can be achieved by a new definition where a 2D matrix plays a role which, in a certain sense we have to explain, is the square root of the matrix in the definition of the two-variable Hermite polynomials.

In section 2, we develop some important relations for powers of linear transformations of 2D vectors with application of Jacobi polynomials which are necessary for the introduction and discussion of Hermite 2D polynomials in section 3 and of Laguerre 2D polynomials in section 4. In section 5, we derive the orthogonality relations. In section 6 we investigate the degenerate case of a vanishing determinant of the transformation. Then we discuss briefly the relations between Hermite and Laguerre 2D polynomials in section 7 before we derive the generating functions for Hermite and Laguerre 2D polynomials in section 8. In section 9, we investigate a specific 'square root problem' for symmetric 2×2 matrices, which is needed to establish the relations between Hermite 2D polynomials and the usual two-variable Hermite polynomials that is made in section 10 and we discuss there the advantages of the new concept of the introduction of Hermite and Laguerre 2D polynomials in comparison to the known concept of two-variable Hermite polynomials.

2. Jacobi polynomials applied to powers of linear transformations of 2D vectors

In this section, we prepare a treatment of powers of linear transformations of 2D vectors that avoids frequent and inconvenient interruptions in the following sections by mathematical problems. This treatment is connected with the application of Jacobi polynomials and we start with their definition and with some of their basic properties.

The Jacobi polynomials $P_j^{(\alpha,\beta)}(u)$ can be defined in the following way [11, 25–27]:

$$P_j^{(\alpha,\beta)}(u) \equiv \frac{(-1)^j}{2^j j!(1-u)^\alpha (1+u)^\beta} \frac{\partial^j}{\partial u^j} \left\{(1-u)^{j+\alpha}(1+u)^{j+\beta}\right\} \tag{2.1}$$

and they possess the following explicit representations [2, 11, 26–28]:

$$\begin{aligned} P_j^{(\alpha,\beta)}(u) &= \sum_{k=0}^{j} \frac{(j+\alpha)!(j+\beta)!}{k!(j+\alpha-k)!(j-k)!(k+\beta)!} \left(\tfrac{1}{2}(u-1)\right)^{j-k} \left(\tfrac{1}{2}(u+1)\right)^{k} \\ &= \frac{(j+\alpha)!}{(j+\alpha+\beta)!} \sum_{l=0}^{j} \frac{(2j+\alpha+\beta-l)!}{l!(j+\alpha-l)!(j-l)!} \left(\tfrac{1}{2}(u-1)\right)^{j-l}. \end{aligned} \tag{2.2}$$

The second form can be obtained from the first one by applying the binomial formula to the decomposition $\{(u+1)/2\}^k = \{(u-1)/2+1\}^k$ and by reordering of the arising double sum with evaluation of one sum. The Jacobi polynomials are available as programmed functions in Stephen Wolfram's *Mathematica*.

The results of the summations in (2.2) are expressible in simple closed form for arbitrary indices (j, α, β) only for the arguments $u = \pm 1$ and in the limiting case $|u| \to \infty$,

$$\begin{aligned} &P_j^{(\alpha,\beta)}(1) = \frac{(j+\alpha)!}{j!\alpha!} \qquad P_j^{(\alpha,\beta)}(-1) = (-1)^j \frac{(j+\beta)!}{j!\beta!} \\ &\lim_{|u|\to\infty} \left(\frac{2}{u}\right)^j P_j^{(\alpha,\beta)}(u) = \frac{(2j+\alpha+\beta)!}{j!(j+\alpha+\beta)!}. \end{aligned} \tag{2.3}$$

The Jacobi polynomials satisfy the following transformation relations which can be proved by using the definition or the explicit representations [28]:

$$
\begin{aligned}
P_j^{(\alpha,\beta)}(u) &= \frac{(j+\alpha)!(j+\beta)!}{j!(j+\alpha+\beta)!}\left(\frac{2}{u-1}\right)^{\alpha} P_{j+\alpha}^{(-\alpha,\beta)}(u) \\
&= \frac{(j+\alpha)!(j+\beta)!}{j!(j+\alpha+\beta)!}\left(\frac{2}{u+1}\right)^{\beta} P_{j+\beta}^{(\alpha,-\beta)}(u) \\
&= \left(\frac{2}{u-1}\right)^{\alpha}\left(\frac{2}{u+1}\right)^{\beta} P_{j+\alpha+\beta}^{(-\alpha,-\beta)}(u) \\
P_j^{(\alpha,\beta)}(u) &= (-1)^j P_j^{(\beta,\alpha)}(-u) \qquad P_j^{(\alpha,\beta)}(u^*) = \left(P_j^{(\alpha,\beta)}(u)\right)^*.
\end{aligned}
\tag{2.4}
$$

For equal upper indices, the Jacobi polynomials are related in a simple way to Gegenbauer (or ultraspherical) polynomials $C_j^{\lambda}(u)$ as follows [11, 22, 26]:

$$
\begin{aligned}
P_j^{(\alpha,\alpha)}(u) &= u^j \sum_{k=0}^{[j/2]} \frac{(j+\alpha)!}{k!(j-2k)!(k+\alpha)!}\left(\frac{u^2-1}{4u^2}\right)^k = \frac{(j+\alpha)!(2\alpha)!}{(j+2\alpha)!\alpha!} C_j^{\alpha+\frac{1}{2}}(u) \\
C_j^{\alpha+\frac{1}{2}}(u) &= \frac{1}{\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{k=0}^{[j/2]} \frac{(-1)^k \Gamma\left(j-k+\alpha+\frac{1}{2}\right)}{k!(j-2k)!}(2u)^{j-2k}.
\end{aligned}
\tag{2.5}
$$

For argument $u = 0$, one obtains from these relations

$$
P_{2l}^{(\alpha,\alpha)}(0) = \frac{(-1)^l(2l+\alpha)!}{2^{2l} l!(l+\alpha)!} \qquad P_{2l+1}^{(\alpha,\alpha)}(0) = 0. \tag{2.6}
$$

This shows that $(-4)^l P_{2l}^{(m-2l,m-2l)}(0)$ are the binomial coefficients $\binom{m}{l} = m!/(l!(m-l)!)$. Some other special values and relations one finds by application of the transformations (2.4) to the written relations.

We now consider general homogeneous linear transformations of 2D vectors and write their components (real or complex ones) in the form of column vectors as follows:

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} U_{11}x_1 + U_{12}x_2 \\ U_{21}x_1 + U_{22}x_2 \end{pmatrix}. \tag{2.7}
$$

The transformation matrix U and its inverse U^{-1} are defined by

$$
U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \qquad U^{-1} = \frac{1}{|U|}\begin{pmatrix} U_{22} & -U_{12} \\ -U_{21} & U_{11} \end{pmatrix} \qquad |U| \equiv U_{11}U_{22} - U_{12}U_{21} \tag{2.8}
$$

where $|U|$ denotes the determinant of U. In many applications (e.g. a lossless beamsplitter, light polarization of a two-mode system) the 2D matrix U is a unitary unimodular matrix $U^{-1} = U^{\dagger}$, $|U| = 1$, but for generality, we do not make such an assumption from the beginning and consider general 2D matrices U including the degenerate case $|U| = 0$. The corresponding transformation of the operators of partial differentiation can be written in the following matrix form with row vectors for the differentiation operators:

$$
\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = \left(\frac{\partial}{\partial x_1'}, \frac{\partial}{\partial x_2'}\right)\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}. \tag{2.9}
$$

It is of basic importance for our further considerations to express the transformations of powers of x_1 and x_2 in the simplest way. By application of the binomial formula and reordering of the sum terms, one finds

$$\begin{aligned} x_1'^m x_2'^n &= (U_{11}x_1 + U_{12}x_2)^m (U_{21}x_1 + U_{22}x_2)^n \\ &= U_{12}^m U_{22}^n \sum_{j=0}^{m+n} x_1^j x_2^{m+n-j} \left(\frac{U_{21}}{U_{22}}\right)^j \sum_{k=0}^{j} \frac{m!n!}{k!(m-k)!(j-k)!(n-j+k)!} \left(\frac{U_{11}U_{22}}{U_{12}U_{21}}\right)^k . \end{aligned} \tag{2.10}$$

This can be written in compact form by using the Jacobi polynomials given in (2.2)

$$\begin{aligned} x_1'^m x_2'^n &= \sum_{j=0}^{m+n} U_{12}^{m-j} U_{22}^{n-j} |U|^j P_j^{(m-j,n-j)}\left(\frac{U_{11}U_{22} + U_{12}U_{21}}{U_{11}U_{22} - U_{12}U_{21}}\right) x_1^j x_2^{m+n-j} \\ &= \left(\sqrt{|U|}\right)^{m+n} \sum_{j=0}^{m+n} \left(\frac{U_{12}}{\sqrt{|U|}}\right)^{m-j} \left(\frac{U_{22}}{\sqrt{|U|}}\right)^{n-j} P_j^{(m-j,n-j)}\left(1 + 2\frac{U_{12}U_{21}}{|U|}\right) x_1^j x_2^{m+n-j}. \end{aligned} \tag{2.11}$$

The identity transformation $U = I$ is connected with the argument $u = 1$ of the Jacobi polynomials in (2.11). The transformation relations (2.4) allow one to write (2.11) in many slightly different forms, for example, in forms where the identity transformation is connected with the argument $u = -1$ in the Jacobi polynomials or where the 'natural' order of the upper indices $(m - j, n - j)$ is reversed into $(n - j, m - j)$. The composition $W = UV$ of two transformations U and V leads to a composition of powers of the components (x_1, x_2) of 2D vectors from which, in connection with relation (2.11), one can derive an addition theorem for the Jacobi polynomials. This is made in appendix A.

One can see from (2.11) that the sum depends only on $U' \equiv U/\sqrt{|U|}$ and that the determinant $|U|$ appears only in a scaling factor $\left(\sqrt{|U|}\right)^{m+n}$ of the transformation. Therefore, for most purposes, it is sufficient to consider only unimodular matrices U' which means

$$|U'| \equiv U'_{11}U'_{22} - U'_{12}U'_{21} = 1. \tag{2.12}$$

The degenerate case of a vanishing determinant $|U| = 0$ makes an exception and has to be considered separately (section 6). For unimodular transformations U', relation (2.11) simplifies according to (the specialization to unimodular matrices is marked in the following by a prime at the corresponding matrices)

$$\begin{aligned} x_1'^m x_2'^n &= \sum_{j=0}^{m+n} U_{12}'^{m-j} U_{22}'^{n-j} P_j^{(m-j,n-j)}\left(1 + 2U'_{12}U'_{21}\right) x_1^j x_2^{m+n-j} \\ &= \sum_{j=0}^{m+n} U_{11}'^{m-j} U_{21}'^{n-j} P_j^{(n-j,m-j)}\left(1 + 2U'_{12}U'_{21}\right) x_1^{m+n-j} x_2^j \qquad |U'| = 1. \end{aligned} \tag{2.13}$$

The argument of the Jacobi polynomials in (2.13) remains, in general, a complex number.

The set of transformations (2.11) with, in general, complex components of the matrix U and with $|U| \neq 0$ forms the two-dimensional complex linear group $GL(2, C)$ with eight independent real parameters. However, we have seen that for most purposes it is sufficient to restrict ourselves without essential loss of generality to the two-dimensional special linear (or unimodular) or symplectic group $SL(2, C) \sim Sp(2, C)$ determined by the additional condition $|U| = 1$ and containing six independent real parameters involved in three complex components of U. An interesting subgroup of $SL(2, C)$ for applications is the two-dimensional special unitary group $SU(2)$ with three real parameters (a lossless beamsplitter and two-mode polarization). Other interesting subgroups of $SL(2, C)$ for applications are the physically different groups $SL(2, R) \sim Sp(2, R) \sim SU(1, 1)$ with three real parameters, where $Sp(2, R)$ denotes the two-dimensional real symplectic group and $SU(1, 1)$ the group of transformations with determinant $|U| = 1$ preserving the indefinite Hermitian form $zz^* - ww^*$ of two complex variables z and w. However, we do not discuss this here in detail.

3. Definition of Hermite 2D polynomials

In this section, we define general Hermite 2D polynomials by means of arbitrary 2D matrices U. They are related to two-variable Hermite polynomials [11], but they are not identical to them and this is taken into account by the new name Hermite 2D polynomials and by the notation. We discuss the relations to two-variable Hermite polynomials in section 10.

Hermite 2D polynomials are sets of polynomials of two independent variables denoted by (x, y) and depending additionally on an arbitrary 2D matrix U as a parameter. If the determinant of U is non-vanishing ($|U| \neq 0$), then the polynomials for the considered U and for the two corresponding unimodular matrices $U' \equiv U/\sqrt{|U|}$ with the two possible signs of the square root are related in a very simple way that makes it possible to restrict oneself in most cases of application or in proofs to unimodular matrices U'. Our aim is a definition with the boundary condition that products of two Hermite polynomials appear as a special case of Hermite 2D polynomials to the 2D unit matrix $U = I$ which means

$$\begin{aligned} H_{m,n}(I; x, y) &\equiv H_m(x)H_n(y) \\ &= \exp\left\{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}(2x)^m(2y)^n \\ &= (-1)^{m+n}\exp\left(x^2 + y^2\right)\frac{\partial^{m+n}}{\partial x^m \partial y^n}\exp\left(-x^2 - y^2\right). \end{aligned} \tag{3.1}$$

We have written here the Hermite polynomials by two different but equivalent definitions from which the first [29, 30] (equation (31)) [31–33] is up to now not as well known as the second one.

We now consider the first of the definitions of the product of two Hermite polynomials in (3.1). Its generalization by the introduction of an arbitrary 2D matrix U can be made in the following way. We first consider the homogeneous linear transformation of the variables (x, y) according to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \Leftrightarrow \quad \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}\right)\begin{pmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{pmatrix} \tag{3.2}$$

and then we define by substitution $(x \to x', y \to y')$ in (3.1)

$$H_{m,n}(U; x, y) \equiv \exp\left\{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}(2x')^m(2y')^n. \tag{3.3}$$

We call the general Hermite 2D polynomials $H_{m,n}(U; x, y)$ simply Hermite 2D polynomials and add in the special case $H_{m,n}(I; x, y) \equiv H_m(x)H_n(y)$, if necessary, the attribute 'special' and call them special Hermite 2D polynomials.

We have now two possibilities to continue starting from definition (3.3). If one expresses (x', y') according to (3.2) then this definition leads to

$$H_{m,n}(U; x, y) = 2^{m+n}\exp\left\{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}(U_{xx}x + U_{xy}y)^m(U_{yx}x + U_{yy}y)^n. \tag{3.4}$$

On the other hand, if one substitutes the second-order partial derivatives in the exponent in (3.3) according to (3.2) then this leads to

$$H_{m,n}(U;x,y) = 2^{m+n}\exp\left\{-\frac{1}{4}\left(\left(U_{xx}^2+U_{xy}^2\right)\frac{\partial^2}{\partial x'^2}+\left(U_{yx}^2+U_{yy}^2\right)\frac{\partial^2}{\partial y'^2}\right.\right.$$
$$\left.\left. +2\left(U_{xx}U_{yx}+U_{xy}U_{yy}\right)\frac{\partial^2}{\partial x'\partial y'}\right)\right\}x'^m y'^n \qquad (3.5)$$

where the arguments (x, y) in the Hermite 2D polynomials $H_{m,n}(U; x, y)$ on the left-hand side may be substituted by $x = (U_{yy}x' - U_{xy}y')/|U|$ and $y = (-U_{yx}x' + U_{xx}y')/|U|$ to show equal variables on both sides. From both formulae (3.4) and (3.5) for the Hermite 2D polynomials, one can derive essentially different basic representations of these polynomials.

The 'disentanglement' of powers $(U_{xx}x + U_{xy}y)^m(U_{yx}x + U_{yy}y)^n$ is connected with the application of Jacobi polynomials $P_j^{(\alpha,\beta)}(u)$ and by using (2.11), one obtains from (3.4)

$$H_{m,n}(U;x,y) = \exp\left\{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right\}\left(\sqrt{|U|}\right)^{m+n}\sum_{j=0}^{m+n}\left(\frac{U_{xy}}{\sqrt{|U|}}\right)^{m-j}\left(\frac{U_{yy}}{\sqrt{|U|}}\right)^{n-j}$$
$$\times P_j^{(m-j,n-j)}\left(1+2\frac{U_{xy}U_{yx}}{|U|}\right)(2x)^j(2y)^{m+n-j}. \qquad (3.6)$$

Now, by using the alternative definition of the Hermite polynomials $H_n(z)$ by application of the operator $\exp(-\frac{1}{4}\partial^2/\partial z^2)$ to $(2z)^n$, one obtains from (3.6) the following first basic representation of the Hermite 2D polynomials:

$$H_{m,n}(U;x,y) = \left(\sqrt{|U|}\right)^{m+n}\sum_{j=0}^{m+n}\left(\frac{U_{xy}}{\sqrt{|U|}}\right)^{m-j}\left(\frac{U_{yy}}{\sqrt{|U|}}\right)^{n-j}$$
$$\times P_j^{(m-j,n-j)}\left(1+2\frac{U_{xy}U_{yx}}{|U|}\right)H_j(x)H_{m+n-j}(y). \qquad (3.7)$$

From this representation or from (3.4), one finds ('*' denotes 'complex conjugation')

$$H_{m,n}(\lambda U;x,y) = \lambda^{m+n}H_{m,n}(U;x,y) \qquad \left(H_{m,n}(U;x,y)\right)^* = H_{m,n}(U^*;x,y)$$
$$H_{m,n}(-U;x,y) = (-1)^{m+n}H_{m,n}(U;x,y) = H_{m,n}(U;-x,-y) \qquad (3.8)$$

where the latter shows that $H_{m,n}(U; x, y)$ possesses the parity $(-1)^{m+n}$. From (3.7), it follows

$$H_{m,n}(U;x,y) = \left(\sqrt{|U|}\right)^{m+n}H_{m,n}\left(U';x,y\right) \qquad U' \equiv \frac{U}{\sqrt{|U|}} \qquad |U'| = 1. \qquad (3.9)$$

This relation shows explicitly that the definition of the Hermite 2D polynomials for an arbitrary matrix U instead of corresponding unimodular matrices U' brings additionally only a factor $\left(\sqrt{|U|}\right)^{m+n}$ in front of the Hermite 2D polynomial. Therefore, for most purposes it is sufficient to restrict oneself to unimodular matrices U' with the exception of the degenerate case $|U| = 0$ which we treat in section 6. We mention here that to an arbitrary matrix U with $|U| \neq 0$ there exist two related unimodular matrices $U' = U/\sqrt{|U|}$ corresponding to the two possible signs of the square root $\sqrt{|U|}$.

Starting from (3.5), we now derive two other interesting representations of the Hermite 2D polynomials which are essentially different in their structure from the representation in

(3.7). The first partial step is to establish the following identity by Taylor series expansion of the exponential operator:

$$\exp\left\{-\frac{U_{xx}U_{yx}+U_{xy}U_{yy}}{2}\frac{\partial^2}{\partial x'\partial y'}\right\}x'^m y'^n$$
$$= \sum_{j=0}^{\{m,n\}}\frac{(-1)^j m!n!}{j!(m-j)!(n-j)!}\left(\tfrac{1}{2}(U_{xx}U_{yx}+U_{xy}U_{yy})\right)^j x'^{m-j}y'^{n-j}. \qquad (3.10)$$

If we now act with the remaining operator in (3.5) on both sides of (3.10), we obtain

$$H_{m,n}(U;x,y) = \left(\sqrt{U_{xx}^2+U_{xy}^2}\right)^m\left(\sqrt{U_{yx}^2+U_{yy}^2}\right)^n$$
$$\times\sum_{j=0}^{\{m,n\}}\frac{(-1)^j m!n!}{j!(m-j)!(n-j)!}\left(2\frac{U_{xx}U_{yx}+U_{xy}U_{yy}}{\sqrt{(U_{xx}^2+U_{xy}^2)(U_{yx}^2+U_{yy}^2)}}\right)^j$$
$$\times H_{m-j}\left(\frac{U_{xx}x+U_{xy}y}{\sqrt{U_{xx}^2+U_{xy}^2}}\right)H_{n-j}\left(\frac{U_{yx}x+U_{yy}y}{\sqrt{U_{yx}^2+U_{yy}^2}}\right) \qquad (3.11)$$

where (x', y') are resubstituted as linear combinations of (x, y) according to (3.2). Thus we have derived in (3.11) the second basic representation of Hermite 2D polynomials which is very different in its structure from the representation in (3.7). Such a structure, as a result of calculations, happens relatively often in two-dimensional problems, in particular, in quantum optics (ordered moments and Fock-state representation of states), whereas then the possible equivalent representation (3.7) is usually unknown.

We now derive a third representation which is nearer to (3.11) than to (3.7). We obtain this representation if we first accomplish in (3.5) the operations leading to Hermite polynomials which we then represent by their explicit form according to

$$\exp\left\{-\frac{1}{4}\left(\left(U_{xx}^2+U_{xy}^2\right)\frac{\partial^2}{\partial x'^2}+\left(U_{yx}^2+U_{yy}^2\right)\frac{\partial^2}{\partial y'^2}\right)\right\}(2x')^m(2y')^n$$
$$=\sum_{k=0}^{[m/2]}\sum_{l=0}^{[n/2]}\frac{(-1)^{k+l}m!n!}{k!(m-2k)!l!(n-2l)!}$$
$$\times\left(U_{xx}^2+U_{xy}^2\right)^k(2x')^{m-2k}\left(U_{yx}^2+U_{yy}^2\right)^l(2y')^{n-2l}. \qquad (3.12)$$

Then, by applying the remaining operator in (3.5) to (3.12) and by using the special Laguerre 2D polynomials $L_{m,n}(z, z^*)$ introduced in [3] (see also (4.1) of the present paper) with variables (x'', y'') instead of (z, z^*), we find the following representation of the Hermite 2D polynomials:

$$H_{m,n}(U;x,y) = \left(\sqrt{2(U_{xx}U_{yx}+U_{xy}U_{yy})}\right)^{m+n}$$
$$\times\sum_{k=0}^{[m/2]}\sum_{l=0}^{[n/2]}\frac{(-1)^{k+l}m!n!}{k!(m-2k)!l!(n-2l)!}\frac{\left(U_{xx}^2+U_{xy}^2\right)^k\left(U_{yx}^2+U_{yy}^2\right)^l}{\left(2\left(U_{xx}U_{yx}+U_{xy}U_{yy}\right)\right)^{k+l}}$$
$$\times L_{m-2k,n-2l}\left(\frac{\sqrt{2}\,(U_{xx}x+U_{xy}y)}{\sqrt{U_{xx}U_{yx}+U_{xy}U_{yy}}},\frac{\sqrt{2}\,(U_{yx}x+U_{yy}y)}{\sqrt{U_{xx}U_{yx}+U_{xy}U_{yy}}}\right). \qquad (3.13)$$

This representation seems to be less successful than (3.11) because here one has a double sum in comparison to a simple sum in (3.11), but it is important to know all the essentially different representations. Taken together, the representations (3.11) and (3.13) show that the Hermite 2D polynomials possess features of a mixing of the structures of the special Hermite 2D and Laguerre 2D polynomials.

Up to now, we have only generalized the first of the two alternative definitions of the special Hermite 2D polynomials $H_{m,n}(I; x, y)$ in (3.1) to the definition of $H_{m,n}(U; x, y)$. Since the definition of $H_{m,n}(U; x, y)$ is now already made, we have to establish the necessary generalization of the second form of the definition of $H_{m,n}(I; x, y)$ in (3.1) in a consistent way. Starting from the explicit form (3.7) of the Hermite 2D polynomials $H_{m,n}(U; x, y)$, we introduce the usual definition of Hermite polynomials as can be seen from (3.1) and by using the transformation of powers of linear combinations of components of 2D vectors in (2.11), we obtain

$$H_{m,n}(U; x, y) = (-1)^{m+n} \exp\left(x^2 + y^2\right) \frac{\partial^{m+n}}{\partial x''^m \partial y''^n} \exp\left(-x^2 - y^2\right). \qquad (3.14)$$

Contrary to (3.2), here we have the following linear transformations of the partial derivatives and of the vector components:

$$\begin{pmatrix} \frac{\partial}{\partial x''} \\ \frac{\partial}{\partial y''} \end{pmatrix} = \begin{pmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \quad \Leftrightarrow \quad (x, y) = (x'', y'') \begin{pmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{pmatrix}. \qquad (3.15)$$

This leads to the following alternative definition of $H_{m,n}(U; x, y)$:

$$\begin{aligned} H_{m,n}(U; x, y) &= (-1)^{m+n} \exp\left(x^2 + y^2\right) \left(U_{xx} \frac{\partial}{\partial x} + U_{xy} \frac{\partial}{\partial y} \right)^m \\ &\quad \times \left(U_{yx} \frac{\partial}{\partial x} + U_{yy} \frac{\partial}{\partial y} \right)^n \exp\left(-x^2 - y^2\right) \\ &= \left\{ U_{xx} \left(2x - \frac{\partial}{\partial x} \right) + U_{xy} \left(2y - \frac{\partial}{\partial y} \right) \right\}^m \\ &\quad \times \left\{ U_{yx} \left(2x - \frac{\partial}{\partial x} \right) + U_{yy} \left(2y - \frac{\partial}{\partial y} \right) \right\}^n 1. \end{aligned} \qquad (3.16)$$

By using (2.11) with the substitutions $x_1 \to 2x - \partial/\partial x, x_2 \to 2y - \partial/\partial y$, the last relation leads to the representation (3.7) of the Hermite 2D polynomials. We can also express (x, y) in (3.14) by (x'', y'') according to (3.15) and come to another form of the definition of Hermite 2D polynomials. However, we find that it does not lead to a new explicit representation not considered up to now and, therefore, we do not write it down.

In this section, we have given two equivalent definitions (3.4) and (3.16) of the Hermite 2D polynomials. From the first definition (3.4), we have derived three different explicit representations of the Hermite 2D polynomials in (3.7), (3.11) and (3.13), where the last two possess a relationship to each other. From the second definition (3.16), we could not derive new explicit representations in comparison to the first but it is necessary to know the structure of all operations which lead to Hermite 2D polynomials and, in this sense, it is as important as the first. The equivalences in the definitions of Hermite 2D polynomials by (3.4) and (3.16) cannot be continued to an operator level. If we consider $H_{m,n}(U; x, y)$ in (3.16) as an operator which does not act on the function 1 as written but on the functions $x^k y^l$ then for $k \neq 0$ or $l \neq 0$ it is no longer equivalent to (3.4) considered as an operator.

4. Definition of Laguerre 2D polynomials

In analogy to the introduction of Hermite 2D polynomials in the preceding section, in this section we define Laguerre 2D polynomials by means of an arbitrary 2D matrix U as a parameter. Our aim is a definition for which the special Laguerre 2D polynomials $L_{m,n}(z, z^*)$ introduced in [3] appear as a special case of the 2D unit matrix $U = I$ which means

$$\begin{aligned} L_{m,n}(I; z, z^*) &\equiv L_{m,n}(z, z^*) \\ &= \exp\left(-\frac{\partial^2}{\partial z \partial z^*}\right) z^m z^{*n} \\ &= (-1)^{m+n} \exp\left(zz^*\right) \frac{\partial^{m+n}}{\partial z^{*m} \partial z^n} \exp\left(-zz^*\right) \\ &= \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m! n!}{j!(m-j)!(n-j)!} z^{m-j} z^{*n-j}. \end{aligned} \tag{4.1}$$

In most (but not all possible) applications, the independent variables are a pair of complex conjugated variables which we have denoted by (z, z^*). In a corresponding way, we now denote the components of the transformation matrix U according to

$$\begin{pmatrix} z' \\ z'^* \end{pmatrix} = \begin{pmatrix} U_{zz} & U_{zz^*} \\ U_{z^*z} & U_{z^*z^*} \end{pmatrix} \begin{pmatrix} z \\ z^* \end{pmatrix} \quad \Leftrightarrow \quad \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z^*}\right) = \left(\frac{\partial}{\partial z'}, \frac{\partial}{\partial z'^*}\right) \begin{pmatrix} U_{zz} & U_{zz^*} \\ U_{z^*z} & U_{z^*z^*} \end{pmatrix}. \tag{4.2}$$

In analogy to (3.4), we now define the Laguerre 2D polynomials in the following way:

$$L_{m,n}(U; z, z^*) \equiv \exp\left(-\frac{\partial^2}{\partial z \partial z^*}\right) z'^m z'^{*n} \tag{4.3}$$

that yields with (z', z'^*) expressed by (z, z^*) according to (4.2)

$$L_{m,n}(U; z, z^*) = \exp\left(-\frac{\partial^2}{\partial z \partial z^*}\right) \left(U_{zz} z + U_{zz^*} z^*\right)^m \left(U_{z^*z} z + U_{z^*z^*} z^*\right)^n. \tag{4.4}$$

If we take $z'^m z'^{*n}$ from (2.11) in a correspondingly rewritten form and if we apply to this the definition (4.1) of the special Laguerre 2D polynomials $L_{m,n}(z, z^*)$, we arrive in analogy to (3.7) at the representation

$$\begin{aligned} L_{m,n}(U; z, z^*) = \left(\sqrt{|U|}\right)^{m+n} \sum_{j=0}^{m+n} \left(\frac{U_{zz^*}}{\sqrt{|U|}}\right)^{m-j} \left(\frac{U_{z^*z^*}}{\sqrt{|U|}}\right)^{n-j} \\ \times P_j^{(m-j,n-j)}\left(1 + 2\frac{U_{zz^*}U_{z^*z}}{|U|}\right) L_{j,m+n-j}(z, z^*). \end{aligned} \tag{4.5}$$

From this first basic representation of the Laguerre 2D polynomials, we find the relations

$$\begin{aligned} &L_{m,n}(\lambda U; z, z^*) = \lambda^{m+n} L_{m,n}(U; z, z^*) \qquad \left(L_{m,n}(U; z, z^*)\right)^* = L_{m,n}(U^*; z^*, z) \\ &L_{m,n}(-U; z, z^*) = (-1)^{m+n} L_{m,n}(U; z, z^*) = L_{m,n}(U; -z, -z^*) \end{aligned} \tag{4.6}$$

where U^* denotes the complex conjugate matrix to U. The last relation shows that the Laguerre 2D polynomials possess the parity $(-1)^{m+n}$. Furthermore, one finds

$$L_{m,n}(U; z, z^*) = \left(\sqrt{|U|}\right)^{m+n} L_{m,n}\left(U'; z, z^*\right) \qquad U' \equiv \frac{U}{\sqrt{|U|}} \qquad |U'| = 1. \tag{4.7}$$

Similarly to the case of Hermite 2D polynomials, the essential information on the Laguerre polynomials is already contained in their definition for unimodular matrices.

A second and a related third basic representation can be obtained if we represent in (4.3) the differential operators in the exponent of the convolution operator by the transformed differential operators with a prime, according to the representation

$$L_{m,n}(U;z,z^*) = \exp\left\{-\left(U_{zz}U_{zz^*}\frac{\partial^2}{\partial z'^2} + U_{z^*z}U_{z^*z^*}\frac{\partial^2}{\partial z'^{*2}} + (U_{zz}U_{z^*z^*} + U_{zz^*}U_{z^*z})\frac{\partial^2}{\partial z'\partial z'^*}\right)\right\} z'^m z'^{*n}. \tag{4.8}$$

If we first accomplish the part of the operations with mixed differentiation operators in the exponent and then the other two parts leading to Hermite polynomials in application to power functions, we obtain in analogy to (3.11),

$$L_{m,n}(U;z,z^*) = \left(\sqrt{U_{zz}U_{zz^*}}\right)^m \left(\sqrt{U_{z^*z}U_{z^*z^*}}\right)^n \times \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!}\left(\frac{U_{zz}U_{z^*z^*} + U_{zz^*}U_{z^*z}}{\sqrt{U_{zz}U_{zz^*}U_{z^*z}U_{z^*z^*}}}\right)^j \times H_{m-j}\left(\frac{U_{zz}z + U_{zz^*}z^*}{2\sqrt{U_{zz}U_{zz^*}}}\right) H_{n-j}\left(\frac{U_{z^*z}z + U_{z^*z^*}z^*}{2\sqrt{U_{z^*z}U_{z^*z^*}}}\right). \tag{4.9}$$

If we first accomplish the operations with squared partial derivatives in the exponent in (4.8) and then the operation with mixed derivatives leading to special Laguerre 2D polynomials, we obtain the representation

$$L_{m,n}(U;z,z^*) = \left(\sqrt{U_{zz}U_{z^*z^*} + U_{zz^*}U_{z^*z}}\right)^{m+n} \times \sum_{k=0}^{[m/2]}\sum_{l=0}^{[n/2]} \frac{(-1)^{k+l}m!n!}{k!(m-2k)!l!(n-2l)!}\frac{(U_{zz}U_{zz^*})^k (U_{z^*z}U_{z^*z^*})^l}{(U_{zz}U_{z^*z^*} + U_{zz^*}U_{z^*z})^{k+l}} \times L_{m-2k,n-2l}\left(\frac{U_{zz}z + U_{zz^*}z^*}{\sqrt{U_{zz}U_{z^*z^*} + U_{zz^*}U_{z^*z}}}, \frac{U_{z^*z}z + U_{z^*z^*}z^*}{\sqrt{U_{zz}U_{z^*z^*} + U_{zz^*}U_{z^*z}}}\right). \tag{4.10}$$

Thus we have obtained three basic representations for the Laguerre 2D polynomials given in (4.5), (4.9) and (4.10). The last representation, although less successful since it contains a double sum in comparison to simple sums of the others, exhibits the close relationship of the Laguerre 2D polynomials $L_{m,n}(U;z,z^*)$ to the special Laguerre 2D polynomials $L_{m,n}(z,z^*)$.

Similarly to the case of Hermite 2D polynomials, there exists a second equivalent definition of the Laguerre 2D polynomials which generalizes the last given definition of $L_{m,n}(z,z^*)$ in (4.1). To obtain this definition, one can start from the explicit representation (4.5) of the Laguerre 2D polynomials and can substitute the special Laguerre 2D polynomials according to the last definition in (4.1). This leads in analogy to (3.15) to the following equivalent definition:

$$L_{m,n}(U;z,z^*) = (-1)^{m+n}\exp\left(zz^*\right)\frac{\partial^{m+n}}{\partial z''^{*m}\partial z''^{n}}\exp\left(-zz^*\right) \tag{4.11}$$

corresponding to the transformations

$$\begin{pmatrix} \frac{\partial}{\partial z''^*} \\ \frac{\partial}{\partial z''} \end{pmatrix} = \begin{pmatrix} U_{zz} & U_{zz^*} \\ U_{z^*z} & U_{z^*z^*} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z^*} \\ \frac{\partial}{\partial z} \end{pmatrix} \quad \Leftrightarrow \quad (z^*, z) = (z''^*, z'') \begin{pmatrix} U_{zz} & U_{zz^*} \\ U_{z^*z} & U_{z^*z^*} \end{pmatrix}.$$

(4.12)

Here the transformation matrix U is connected with the transformation of the row vectors (z^*, z) with reversed order of the components in comparison to (4.2). That this is correct can be seen if one writes (4.11) in the following more detailed form:

$$\begin{aligned} L_{m,n}(U; z, z^*) &= (-1)^{m+n} \exp\left(zz^*\right) \left(U_{zz}\frac{\partial}{\partial z^*} + U_{zz^*}\frac{\partial}{\partial z}\right)^m \\ &\quad \times \left(U_{z^*z}\frac{\partial}{\partial z^*} + U_{z^*z^*}\frac{\partial}{\partial z}\right)^n \exp\left(-zz^*\right) \\ &= \left\{U_{zz}\left(z - \frac{\partial}{\partial z^*}\right) + U_{zz^*}\left(z^* - \frac{\partial}{\partial z}\right)\right\}^m \\ &\quad \times \left\{U_{z^*z}\left(z - \frac{\partial}{\partial z^*}\right) + U_{z^*z^*}\left(z^* - \frac{\partial}{\partial z}\right)\right\}^n 1. \end{aligned} \tag{4.13}$$

The connection of the matrix U with (z, z^*) appears here again in the 'right' order.

In analogy to the Hermite 2D polynomials, we have two equivalent basic definitions of the Laguerre 2D polynomials given in (4.4) and (4.13) leading to three essentially different explicit representations given in (4.5), (4.9) and (4.10).

5. Hermite and Laguerre 2D functions and their orthonormalization

Besides the Hermite 2D polynomials $H_{m,n}(U; x, y)$, we introduce Hermite 2D functions $h_{m,n}(U; x, y)$ in the following way:

$$h_{m,n}(U; x, y) \equiv \frac{1}{\sqrt{\pi}} \exp\left(-\tfrac{1}{2}(x^2 + y^2)\right) \frac{H_{m,n}(U; x, y)}{\sqrt{2^{m+n} m! n!}}. \tag{5.1}$$

The main purpose of the introduction of the Hermite 2D functions $h_{m,n}(U; x, y)$ in the above form is their orthonormalization and completeness which we derive below.

The relations (3.8) and (3.9) can be continued to the Hermite 2D functions by the formal substitution $H \to h$. This shows that $h_{m,n}(U; x, y)$ is essentially defined already by the corresponding unimodular matrices $U' \equiv U/\sqrt{|U|}$. The following explicit representation:

$$\begin{aligned} h_{m,n}(U; x, y) &= \frac{\left(\sqrt{|U|}\right)^{m+n}}{\sqrt{m!n!}} \sum_{j=0}^{m+n} \sqrt{j!(m+n-j)!} \left(\frac{U_{xy}}{\sqrt{|U|}}\right)^{m-j} \left(\frac{U_{yy}}{\sqrt{|U|}}\right)^{n-j} \\ &\quad \times P_j^{(m-j,n-j)}\left(1 + 2\frac{U_{xy}U_{yx}}{|U|}\right) h_j(x) h_{m+n-j}(y) \end{aligned} \tag{5.2}$$

is obtained from (5.1) by using (3.7), where $h_n(x)$ are Hermite functions defined by [2]

$$h_n(x) \equiv \frac{1}{\pi^{1/4}} \exp\left(-\tfrac{1}{2}(x^2)\right) \frac{H_n(x)}{\sqrt{2^n n!}} \qquad \int_{-\infty}^{+\infty} \mathrm{d}x\, h_m(x) h_n(x) = \delta_{m,n}. \tag{5.3}$$

From the representation (5.2) it becomes obvious that the Hermite 2D functions obey the eigenvalue equation for a degenerate 2D harmonic oscillator to the eigenvalue $m + n + 1$ since the functions $h_j(x)h_{m+n-j}(y)$ for arbitrary $j = 0, 1, \ldots, m + n$ obey such an equation (see [2]). Thus we have

$$\left\{\frac{x^2 + y^2}{2} - \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\} h_{m,n}(U; x, y) = (m + n + 1) h_{m,n}(U; x, y). \tag{5.4}$$

For fixed U, the eigenvalue $m+n+1$ is $(m+n+1)$-fold degenerate since all $h_{m,n}(U; x, y)$ with given sum $m+n$ possess the same eigenvalue to the operator of this equation. To discriminate between $h_{m,n}(U; x, y)$ with equal sum $m+n$ but different (m, n), we need a second independent operator to which $h_{m,n}(U; x, y)$ are eigensolutions. We do not treat this here with the consequence that we cannot derive the orthogonality relations alone from this eigenvalue equation, but instead of this we find this directly from the explicit representation. However, from the eigenvalue equation (5.4), one can see the following. The eigenvalues of the operator of this equation do not depend on the matrix U. Therefore, between the two sets of $(m+n+1)$ functions $h_{m,n}(U; x, y)$ and $h_{k,l}(V; x, y)$ with fixed sum $m+n = k+l$ but arbitrary matrices U and V have to exist transformation relations, whereas $h_{m,n}(U; x, y)$ and $h_{k,l}(V; x, y)$ with $m+n \neq k+l$ have to be orthogonal. The transformation relations between $h_{m,n}(U; x, y)$ and the special set $h_{j,m+n-j}(I; x, y)$, $(j = 0, 1, \ldots, m+n)$ can be taken from (3.7) explicitly in connection with (5.1).

The derivation of the orthonormalization can be made for simplicity first for unimodular matrices and then it can be extended in an easy way to the general case. We consider for this purpose the following integral over the product of two Hermite 2D functions to arbitrary unimodular matrices U' and V' and use the orthonormality of the usual Hermite functions $h_n(x)$ as follows ($|U'| = |V'| = 1$):

$$\begin{aligned}\int \mathrm{d}x \wedge \mathrm{d}y\, h_{k,l}(V'; x, y) h_{m,n}(U'; x, y) &= \frac{1}{\pi\sqrt{2^{k+l+m+n} k!l!m!n!}} \\ &\quad \times \sum_{i=0}^{k+l} V_{xy}'^{k-i} V_{yy}'^{l-i} P_i^{(k-i,l-i)}\left(1 + 2V_{xy}'V_{yx}'\right) \\ &\quad \times \sum_{j=0}^{m+n} U_{xy}'^{m-j} U_{yy}'^{n-j} P_j^{(m-j,n-j)}\left(1 + 2U_{xy}'U_{yx}'\right) \\ &\quad \times \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-x^2\right) H_i(x) H_j(x) \int_{-\infty}^{+\infty} \mathrm{d}y \exp\left(-y^2\right) H_{k+l-i}(y) H_{m+n-j}(y) \\ &= \delta_{k+l,m+n} \frac{1}{\sqrt{k!l!m!n!}} \sum_{j=0}^{m+n} j!(m+n-j)! U_{xy}'^{m-j} U_{yy}'^{n-j} V_{xy}'^{k-j} V_{yy}'^{m+n-k-j} \\ &\quad \times P_j^{(m-j,n-j)}\left(1 + 2U_{xy}'U_{yx}'\right) P_j^{(k-j,m+n-k-j)}\left(1 + 2V_{xy}'V_{yx}'\right). \end{aligned} \tag{5.5}$$

By using relation (A.4) of appendix A, one can write (5.5) in the form

$$\begin{aligned}\int \mathrm{d}x \wedge \mathrm{d}y\, h_{k,l}(V'; x, y) h_{m,n}(U'; x, y) \\ = \delta_{k+l,m+n} \sqrt{\frac{k!l!}{m!n!}} W_{xy}'^{m-k} W_{yy}'^{n-k} P_k^{(m-k,n-k)}(1 + 2W_{xy}'W_{yx}')\end{aligned} \tag{5.6}$$

where $(W_{xx}', W_{xy}', W_{yx}', W_{yy}')$ are the components of the unimodular product matrix $W' = U'V'$. If $V' = U'^{-1}$ or $W' = U'V' = I$ is the unit matrix and therefore $W_{xx}' = W_{yy}' = 1$, $W_{xy}' = W_{yx}' = 0$, one obtains with the special value of the Jacobi polynomials $P_j^{(\alpha,\beta)}(u)$ for argument $u = 1$ given in (2.3)

$$\int \mathrm{d}x \wedge \mathrm{d}y\, h_{k,l}(U'^{-1}; x, y) h_{m,n}(U'; x, y) = \delta_{k,m}\delta_{l,n}. \tag{5.7}$$

This basic orthonormality relation for Hermite 2D functions is not only true for unimodular matrices U' for which it was derived but also for general matrices U as we will now show.

For general U, V and $W = UV$, one has to add in the right-hand side the factor $\left(\sqrt{|U|}\right)^{m+n}\left(\sqrt{|V|}\right)^{k+l}$ and, due to the presence of the Kronecker symbol on this side, relation (5.6) generalizes to

$$\int \mathrm{d}x \wedge \mathrm{d}y\, h_{k,l}(V; x, y) h_{m,n}(U; x, y)$$
$$= \delta_{k+l,m+n}\sqrt{\frac{k!l!}{m!n!}}\,(|W|)^k\, W_{xy}^{m-k} W_{yy}^{n-k} P_k^{(m-k,n-k)}\left(1 + 2\frac{W_{xy}W_{yx}}{|W|}\right). \tag{5.8}$$

By setting $V = U^{-1}$ and therefore $W = UV = I$ in this relation or directly by generalization of (5.7), one finds

$$\int \mathrm{d}x \wedge \mathrm{d}y\, h_{k,l}(U^{-1}; x, y) h_{m,n}(U; x, y) = \delta_{k,m}\delta_{l,n}. \tag{5.9}$$

This means that the basic orthonormality relations (5.7) remain unchanged by the transition from unimodular to general 2D matrices U.

The sets of functions $h_{m,n}(U; x, y)$ for arbitrary U form complete sets of functions of two variables (x, y) since they can be obtained by continuous and reversible transformations from the set of special Hermite 2D functions $h_{m,n}(I; x, y) = h_m(x)h_n(y)$ from which their completeness is known (proof, for example, by the Mehler formula, see [3]). Therefore, it is not necessary to prove anew the fact of completeness and together with the orthonormality (5.9), we can give immediately the rigorous formulation of the completeness relation

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{m,n}(U; x, y) h_{m,n}(U^{-1}; x', y') = \delta(x - x')\delta(y - y'). \tag{5.10}$$

This leads to the following possible expansions of functions $f(x, y)$:

$$f(x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{m,n} h_{m,n}(U; x, y) \qquad c_{m,n} = \int \mathrm{d}x \wedge \mathrm{d}y\, h_{m,n}(U^{-1}; x, y) f(x, y) \tag{5.11}$$

where the integration goes over the whole (x, y)-plane. The 2D matrix U is herein a free parameter which can be chosen in an appropriate way. Exactly speaking, one has to determine the spaces of functions $f(x, y)$ for which all coefficients $c_{m,n}$ remain finite and lead to convergent expansions but we will not do so. We only mention that these spaces are isomorphic to a realization of the Fock space for a two-mode system and are therefore Hilbert spaces but for some problems, certainly, one can use extensions of these spaces to rigged Hilbert spaces and can use the expansions (5.11) in the sense of weak convergence of generalized functions.

For the purpose of orthonormalization of the Laguerre 2D polynomials $L_{m,n}(U; z, z^*)$, we now define Laguerre 2D functions $l_{m,n}(U; z, z^*)$ by

$$l_{m,n}(U; z, z^*) \equiv \frac{1}{\sqrt{\pi}} \exp\left(-\tfrac{1}{2}zz^*\right) \frac{L_{m,n}(U; z, z^*)}{\sqrt{m!n!}}. \tag{5.12}$$

The discussion is analogous in many regards to the discussion for Hermite 2D functions and we do not repeat it completely. The special case of the unit matrix $U = I$ leads to the special Laguerre 2D functions $l_{m,n}(z, z^*)$ introduced and discussed in [2]

$$l_{m,n}(z, z^*) = \frac{1}{\sqrt{\pi}} \exp\left(-\tfrac{1}{2}zz^*\right) \frac{1}{\sqrt{m!n!}} \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!} z^{m-j} z^{*\,n-j}. \tag{5.13}$$

They satisfy the following orthonormality relations [2] (see also appendix B; note that the 2-form $(\mathrm{i}/2)\,\mathrm{d}z\wedge\mathrm{d}z^* = \mathrm{d}x\wedge\mathrm{d}y$ is the area element of the complex plane):

$$\int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}z\wedge\mathrm{d}z^*\, l_{k,l}(z^*,z)l_{m,n}(z,z^*) = \delta_{k,m}\delta_{l,n}. \tag{5.14}$$

By using these special Laguerre 2D functions, one can represent (5.11) in the following explicit form in analogy to (5.2) for Hermite functions:

$$l_{m,n}(U;z,z^*) = \frac{\left(\sqrt{|U|}\right)^{m+n}}{\sqrt{m!n!}}\sum_{j=0}^{m+n}\sqrt{j!(m+n-j)!}\left(\frac{U_{zz^*}}{\sqrt{|U|}}\right)^{m-j}\left(\frac{U_{z^*z^*}}{\sqrt{|U|}}\right)^{n-j}$$
$$\times P_j^{(m-j,n-j)}\left(1+2\frac{U_{zz^*}U_{z^*z}}{|U|}\right) l_{j,m+n-j}(z,z^*). \tag{5.15}$$

Since for arbitrary $j = 0,1,\ldots,m+n$, $l_{j,m+n-j}(z,z^*)$ obey the eigenvalue equation for a degenerate 2D harmonic oscillator to the eigenvalue $m+n+1$ (see [2]), the functions $l_{m,n}(U;z,z^*)$ obey the same eigenvalue equation which is

$$\left(\frac{zz^*}{2} - 2\frac{\partial^2}{\partial z\partial z^*}\right) l_{m,n}(U;z,z^*) = (m+n+1)l_{m,n}(U;z,z^*). \tag{5.16}$$

This equation is one reason for the introduction of this set of functions. The other reason is that $l_{m,n}(U;z,z^*)$ obey orthonormality relations which we now derive.

The derivation of orthonormality relations is completely analogous to (5.5) and we make use of this. Therefore, we obtain in analogy to (5.8)

$$\int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}z\wedge\mathrm{d}z^*\, l_{k,l}(V;z^*,z)l_{m,n}(U;z,z^*)$$
$$= \delta_{k+l,m+n}\sqrt{\frac{k!l!}{m!n!}}\,(|W|)^k\, W_{zz^*}^{m-k} W_{z^*z^*}^{n-k} P_k^{(m-k,n-k)}\left(1+2\frac{W_{zz^*}W_{z^*z}}{|W|}\right) \tag{5.17}$$

from which in the special case $V = U^{-1}$ and therefore $W = UV = I$ it follows

$$\int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}z\wedge\mathrm{d}z^*\, l_{k,l}(U^{-1};z^*,z)l_{m,n}(U;z,z^*) = \delta_{k,m}\delta_{l,n}. \tag{5.18}$$

The completeness relation in analogy to (5.10) is

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} l_{m,n}(U;z,z^*)l_{m,n}(U^{-1};z'^*,z') = \delta(z-z',z^*-z'^*) \tag{5.19}$$

and the expansion of an arbitrary function $f(z,z^*)$ reads

$$f(z,z^*) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{m,n}l_{m,n}(U;z,z^*) \qquad c_{m,n} = \int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}z\wedge\mathrm{d}z^* l_{m,n}(U^{-1};z^*,z)f(z,z^*). \tag{5.20}$$

One can use (4.6) to represent (5.17)–(5.20) in slightly different forms.

6. Degenerate case of a vanishing determinant

We now consider the degenerate case of a vanishing determinant of U in the Hermite 2D and Laguerre 2D polynomials. In this case the corresponding polynomials simplify considerably. We begin with the Hermite 2D polynomials.

A vanishing determinant of U

$$|U| = U_{xx}U_{yy} - U_{xy}U_{yx} = 0 \tag{6.1}$$

is equivalent to the linear dependence of the two lines of the matrix U and leads to a linear dependence of x' and y' in (3.2) according to

$$x' = U_{xx}x + U_{xy}y = \frac{U_{xx}}{U_{yx}}y' \qquad y' = U_{yx}x + U_{yy}y = \frac{U_{yx}}{U_{xx}}x'. \tag{6.2}$$

The definition (3.4) of the Hermite 2D polynomials can now be written as follows with a result which again can be taken from (3.4):

$$\begin{aligned}(H_{m,n}(U; x, y))_{|U|=0} &= \left(\frac{U_{yx}}{U_{xx}}\right)^n \exp\left\{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}(2(U_{xx}x + U_{yx}y))^{m+n} \\ &= \left(\frac{U_{yx}}{U_{xx}}\right)^n H_{m+n,0}(U; x, y).\end{aligned} \tag{6.3}$$

A similar second variant relates the result to $H_{0,m+n}(U; x, y)$ instead of $H_{m+n,0}(U; x, y)$. If we take these special Hermite 2D polynomials in their explicit form from (3.11), we obtain

$$\begin{aligned}(H_{m,n}(U; x, y))_{|U|=0} &= \left(\frac{U_{yx}}{U_{xx}}\right)^n \left(\sqrt{U_{xx}^2 + U_{xy}^2}\right)^{m+n} H_{m+n}\left(\frac{U_{xx}x + U_{xy}y}{\sqrt{U_{xx}^2 + U_{xy}^2}}\right) \\ &= \left(\frac{U_{xy}}{U_{yy}}\right)^m \left(\sqrt{U_{yx}^2 + U_{yy}^2}\right)^{m+n} H_{m+n}\left(\frac{U_{yx}x + U_{yy}y}{\sqrt{U_{yx}^2 + U_{yy}^2}}\right).\end{aligned} \tag{6.4}$$

Hence in the degenerate case, the Hermite 2D polynomials lead to usual Hermite polynomials of the sum of the indices and with arguments in the form of linear combinations of x and y and with some factors in front of the polynomials.

If we first apply the binomial formula in the representation in (6.3) and if we then apply the operator to the result, we obtain the following alternative representations:

$$\begin{aligned}(H_{m,n}(U; x, y))_{|U|=0} &= \left(\frac{U_{yx}}{U_{xx}}\right)^n \sum_{j=0}^{m+n} \frac{(m+n)!}{j!(m+n-j)!} U_{xx}^j U_{xy}^{m+n-j} H_j(x) H_{m+n-j}(y) \\ &= \left(\frac{U_{xx}}{U_{yx}}\right)^m \sum_{j=0}^{m+n} \frac{(m+n)!}{j!(m+n-j)!} U_{yx}^j U_{yy}^{m+n-j} H_j(x) H_{m+n-j}(y).\end{aligned} \tag{6.5}$$

This can also be derived from (3.7) by a limiting procedure using (B.8) of appendix B. By combining (6.4) and (6.5), we obtain an identity which is the addition theorem for the Hermite polynomials [11, 27]. Other forms of the Hermite 2D polynomials in the degenerate case can be taken from (3.11) and (3.13). In particular, one obtains from (3.11) (we write down only one of the possible two variants)

$$(H_{m,n}(U;x,y))_{|U|=0} = \left(\frac{U_{yx}}{U_{xx}}\right)^n \left(\sqrt{U_{xx}^2+U_{xy}^2}\right)^{m+n} \sum_{j=0}^{\{m,n\}} \frac{(-2)^j m!n!}{j!(m-j)!(n-j)!}$$

$$\times H_{m-j}\left(\frac{U_{xx}x+U_{xy}y}{\sqrt{U_{xx}^2+U_{xy}^2}}\right) H_{n-j}\left(\frac{U_{xx}x+U_{xy}y}{\sqrt{U_{xx}^2+U_{xy}^2}}\right). \tag{6.6}$$

This representations compared with (6.4) reveals a further known identity for Hermite polynomials [11] ((10.13), equation (36)), however, not in its simplest form. The given different representations are important for recognizing them as the degenerate case of Hermite 2D polynomials if they happen as a result of calculation.

We now consider the vanishing of the determinant of U in the case of the Laguerre 2D polynomials which means

$$|U| = U_{zz}U_{z^*z^*} - U_{zz^*}U_{z^*z} = 0. \tag{6.7}$$

In analogy to (6.3), the definition (4.4) can now be written as

$$\begin{aligned}(L_{m,n}(U;z,z^*))_{|U|=0} &= \left(\frac{U_{z^*z}}{U_{zz}}\right)^n \exp\left(-\frac{\partial^2}{\partial z\partial z^*}\right)\left(U_{zz}z+U_{zz^*}z^*\right)^{m+n} \\ &= \left(\frac{U_{z^*z}}{U_{zz}}\right)^n L_{m+n,0}(U;z,z^*)\end{aligned} \tag{6.8}$$

and a similar second variant is possible. With the explicit form of $L_{m+n,0}(U;z,z^*)$ taken from (4.9), we find the explicit form

$$(L_{m,n}(U;z,z^*))_{|U|=0} = \left(\frac{U_{z^*z}}{U_{zz}}\right)^n \left(\sqrt{U_{zz}U_{zz^*}}\right)^{m+n} H_{m+n}\left(\frac{U_{zz}z+U_{zz^*}z^*}{2\sqrt{U_{zz}U_{zz^*}}}\right). \tag{6.9}$$

If we first apply in (6.8) the binomial formula and then the operator to the result, we obtain the following representation by special Laguerre 2D polynomials:

$$(L_{m,n}(U;z,z^*))_{|U|=0} = \left(\frac{U_{z^*z}}{U_{zz}}\right)^n \sum_{j=0}^{m+n} \frac{(m+n)!}{j!(m+n-j)!} U_{zz}^j U_{zz^*}^{m+n-j} L_{j,m+n-j}(z,z^*). \tag{6.10}$$

Furthermore, the following representation can be obtained from (4.9):

$$(L_{m,n}(U;z,z^*))_{|U|=0} = \left(\frac{U_{z^*z}}{U_{zz}}\right)^n \left(\sqrt{U_{zz}U_{zz^*}}\right)^{m+n} \sum_{j=0}^{\{m,n\}} \frac{(-2)^j m!n!}{j!(m-j)!(n-j)!}$$

$$\times H_{m-j}\left(\frac{U_{zz}z+U_{zz^*}z^*}{2\sqrt{U_{zz}U_{zz^*}}}\right) H_{n-j}\left(\frac{U_{zz}z+U_{zz^*}z^*}{2\sqrt{U_{zz}U_{zz^*}}}\right). \tag{6.11}$$

In (6.9), (6.10) and (6.11), we did not write down the second possible variant. By comparison of the right-hand sides of relations (6.9)–(6.11), we obtain three identities from which one identity for Hermite polynomials is known [11] (again (10.13), equation (36)).

7. Relations of Hermite 2D to Laguerre 2D polynomials

In this section, we establish the relations between Hermite 2D and Laguerre 2D polynomials. We start from the relations between the two pairs of variables (x, y) and (z, z^*) which we write in the following matrix form:

$$\begin{aligned}
&\begin{pmatrix} z \\ z^* \end{pmatrix} = (1-\mathrm{i})Z \begin{pmatrix} x \\ y \end{pmatrix} \qquad Z \equiv \begin{pmatrix} \frac{1}{2}(1+\mathrm{i}) & -\frac{1}{2}(1-\mathrm{i}) \\ \frac{1}{2}(1+\mathrm{i}) & \frac{1}{2}(1-\mathrm{i}) \end{pmatrix} \qquad |Z| = 1 \\
&\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{2}(1+\mathrm{i})\, Z^{-1} \begin{pmatrix} z \\ z^* \end{pmatrix} \qquad Z^{-1} = \begin{pmatrix} \frac{1}{2}(1-\mathrm{i}) & \frac{1}{2}(1-\mathrm{i}) \\ -\frac{1}{2}(1+\mathrm{i}) & \frac{1}{2}(1+\mathrm{i}) \end{pmatrix} = Z^{\dagger}.
\end{aligned} \tag{7.1}$$

We have split factors in these transformations in such a way that the remaining transformation matrix Z becomes a unimodular one ($|Z| = 1$) and, in addition, it became a unitary matrix ($Z^{-1} = Z^{\dagger}$). For the product of Z with the transposed matrix Z^{T}, we find

$$ZZ^{\mathsf{T}} = \mathrm{i}\sigma_1 \quad \Leftrightarrow \quad (1-\mathrm{i})^2 ZZ^{\mathsf{T}} = 2\sigma_1 \qquad \sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.2}$$

where σ_1 is the first of the three Pauli spin matrices. We use this relation later in the derivation of the generating functions for the Laguerre 2D polynomials.

The transformations from $L_{m,n}(U; z, z^*)$ to $H_{m,n}(V; x, y)$ and vice versa have to take into account the additional transformations of the variables (x, y) into (z, z^*) given in (7.1) before defining the corresponding polynomials and one obtains

$$\begin{aligned}
L_{m,n}(U; x+\mathrm{i}y, x-\mathrm{i}y) &= \left(\tfrac{1}{2}(1-\mathrm{i})\right)^{m+n} H_{m,n}\,(UZ; x, y) \\
H_{m,n}(U; x, y) &= (1+\mathrm{i})^{m+n}\, L_{m,n}\big(UZ^{-1}; x+\mathrm{i}y, x-\mathrm{i}y\big).
\end{aligned} \tag{7.3}$$

The corresponding relations between the Hermite and Laguerre 2D functions are

$$\begin{aligned}
l_{m,n}(U; x+\mathrm{i}y, x-\mathrm{i}y) &= \left(\frac{1-\mathrm{i}}{\sqrt{2}}\right)^{m+n} h_{m,n}\,(UZ; x, y) \qquad \frac{1-\mathrm{i}}{\sqrt{2}} = \exp\left(-\tfrac{1}{4}\mathrm{i}\pi\right) \\
h_{m,n}(U; x, y) &= \left(\frac{1+\mathrm{i}}{\sqrt{2}}\right)^{m+n} l_{m,n}\big(UZ^{-1}; x+\mathrm{i}y, x-\mathrm{i}y\big) \qquad \frac{1+\mathrm{i}}{\sqrt{2}} = \exp\left(\tfrac{1}{4}\mathrm{i}\pi\right).
\end{aligned} \tag{7.4}$$

We see that the Hermite and Laguerre 2D polynomials as well as 2D functions are closely related by changing the matrix U into UZ or UZ^{-1}, correspondingly. Nevertheless, it would be unfavourable to renounce the definition of one kind of these polynomials or functions because the Hermite 2D polynomials and functions are more suited for the representation by real variables and the Laguerre 2D polynomials and functions to the representation by complex variables. In relations (7.4) for the 2D functions we have phase factors which are $(m+n)$th powers of $\exp(\pm\mathrm{i}\pi/4)$ and which could be included in the definition of the Laguerre 2D functions that, however, seems to be inconvenient. In relations (7.3) for the polynomials we have, additionally, positive or negative $(m+n)$th powers of $\sqrt{2}$, but it is also unfavourable to include them in the definition of the Laguerre 2D polynomials because then many relations for these polynomials take on an inappropriately complicated form.

The special case $U = I$ in (7.3) and (7.4) provides the relations between special Hermite and Laguerre 2D polynomials or functions, for example, in connection with (3.7) and (4.5)

$$
\begin{aligned}
L_{m,n}(x+\mathrm{i}y, x-\mathrm{i}y) &= \left(\tfrac{1}{2}(1-\mathrm{i})\right)^{m+n} H_{m,n}(Z; x, y) \\
&= (-1)^n \left(\tfrac{1}{2}\mathrm{i}\right)^{m+n} \sum_{j=0}^{m+n} (-\mathrm{i}2)^j P_j^{(m-j,n-j)}(0) H_j(x) H_{m+n-j}(y) \\
H_m(x)H_n(y) &= (1+\mathrm{i})^{m+n} L_{m,n}\left(Z^{-1}; x+\mathrm{i}y, x-\mathrm{i}y\right) \\
&= \mathrm{i}^n \sum_{j=0}^{m+n} 2^j P_j^{(m-j,n-j)}(0) L_{j,m+n-j}(x+\mathrm{i}y, x-\mathrm{i}y).
\end{aligned}
\tag{7.5}
$$

These relations are already given in [2, 3] in a slightly different form related by the transformation relations (2.4) of the Jacobi polynomials with (7.5). We mention that the argument of the Jacobi polynomials in the special relations (7.5) is equal to zero.

8. Generating functions of Hermite and Laguerre 2D polynomials

We now derive the simplest generating functions for the Hermite and Laguerre 2D polynomials. This enables us to establish the relation to the two-variable Hermite polynomials.

The simplest generating function is connected with the following sum over Hermite 2D polynomials with two parameters (s, t) which by using the definition (3.4) yields

$$
\begin{aligned}
\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{s^m t^n}{m!n!} H_{m,n}(U; x, y) &= \exp\left\{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\} \\
&\quad\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!n!} \left(2s(U_{xx}x + U_{xy}y)\right)^m \left(2t(U_{yx}x + U_{yy}y)\right)^n \\
&= \exp\left\{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\} \exp\left\{2(sU_{xx} + tU_{yx})x + 2(sU_{xy} + tU_{yy})y\right\} \\
&= \exp\left\{2(sU_{xx} + tU_{yx})x + 2(sU_{xy} + tU_{yy})y \right. \\
&\quad \left. -(sU_{xx} + tU_{yx})^2 - (sU_{xy} + tU_{yy})^2\right\}.
\end{aligned}
\tag{8.1}
$$

The result of the application of the exponential operator in (8.1) to the exponential function can be obtained by Taylor series expansion of this operator. This result possesses an interesting algebraic structure which is clarified by the following relations:

$$
\begin{aligned}
&(sU_{ss} + tU_{yx})^2 + (sU_{xy} + tU_{yy})^2 \\
&\qquad = (s, t)\begin{pmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{pmatrix}\begin{pmatrix} U_{xx} & U_{yx} \\ U_{xy} & U_{yy} \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = \bar{s}UU^{\mathsf{T}}\bar{s} \\
&(sU_{xx} + tU_{yx})x + (sU_{xy} + tU_{yy})y = (s, t)\begin{pmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \bar{s}U\bar{x}
\end{aligned}
\tag{8.2}
$$

with the following abbreviations for 2D vectors $\bar{x}$ and $\bar{s}$:

$$
\bar{x} \equiv (x, y) \qquad \bar{s} \equiv (s, t) \qquad \text{or} \qquad \bar{x} \equiv \begin{pmatrix} x \\ y \end{pmatrix} \qquad \bar{s} \equiv \begin{pmatrix} s \\ t \end{pmatrix}. \tag{8.3}
$$

We do not distinguish by a symbol for transposition between the row and corresponding column vectors because their meaning becomes clear from their position in the bilinear or quadratic forms where they are involved. The matrix U^{T} is the transposed matrix to the matrix U. With this notation, we can write the generating function (8.1) in the following form:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m t^n}{m!n!}H_{m,n}(U;x,y) = \exp\left(2\bar{s}U\bar{x} - \bar{s}UU^{\mathsf{T}}\bar{s}\right). \tag{8.4}$$

This is in great analogy to the generating functions for the usual Hermite polynomials if we denote $\bar{s}U = U^{\mathsf{T}}\bar{s} \equiv s$ in the one-dimensional case.

The simplest generating function for the Laguerre 2D polynomials can be obtained from the generating function (8.4) for Hermite 2D polynomials by simple substitutions via relations (7.1)–(7.3) according to

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m t^n}{m!n!}L_{m,n}(U;z,z^*) &= \exp\left\{2\left(\tfrac{1}{2}(1-\mathrm{i})\right)\bar{s}UZ\bar{x} - \left(\tfrac{1}{2}(1-\mathrm{i})\right)^2\bar{s}UZZ^{\mathsf{T}}U^{\mathsf{T}}\bar{s}\right\}\\ &= \exp\left(\bar{s}U\bar{z} - \tfrac{1}{2}\bar{s}U\sigma_1U^{\mathsf{T}}\bar{s}\right)\end{aligned} \tag{8.5}$$

with the abbreviations in analogy to (8.3)

$$\bar{z} \equiv (z, z^*) \qquad \bar{s} \equiv (s, t) \qquad \text{or} \qquad \bar{z} \equiv \begin{pmatrix} z \\ z^* \end{pmatrix} \qquad \bar{s} \equiv \begin{pmatrix} s \\ t \end{pmatrix} \tag{8.6}$$

and with σ_1 as the first of the Pauli spin matrices. The generating function (8.5) can also be derived in analogy to (8.1) by using the definition (4.4) of the Laguerre 2D polynomials.

The generating function (8.4) for the Hermite 2D polynomials suggests the following ν-dimensional generalization to Hermite νD polynomials in the rank of a definition

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_\nu=0}^{\infty}\frac{s_1^{n_1}\cdots s_\nu^{n_\nu}}{n_1!\cdots n_\nu!}H_{n_1,\ldots,n_\nu}(U;x_1,\ldots,x_\nu) = \exp\left(2sUx - sUU^{\mathsf{T}}s\right) \tag{8.7}$$

with the abbreviations

$$x \equiv (x_1,\ldots,x_\nu) \qquad s \equiv (s_1,\ldots,s_\nu) \qquad U = \begin{pmatrix} U_{11} & \ldots & U_{1\nu} \\ \vdots & & \vdots \\ U_{\nu 1} & \ldots & U_{\nu\nu} \end{pmatrix} \tag{8.8}$$

and with the corresponding transposed vectors and matrix U^{T}. The direct definitions (3.4) and (3.5) can be easily generalized in agreement with the generating function (8.7) but there arise new problems to obtain explicit representations in analogy to the 2D case. It is obvious that (8.7) is the most natural generalization of the Hermite 2D to Hermite νD polynomials but, contrary to the 2D case, applications are rarely found for $\nu \geqslant 3$ up to now. We mention that in the 1D case we obtain $H_n(U;x) = U^n H_n(x)$ with scalar U and the introduction of $H_n(U;x)$ becomes superfluous.

9. Square roots of symmetric 2 × 2 matrices

In preparation to establish the relation of Hermite 2D polynomials to usual two-variable Hermite polynomials, we consider a specific problem which is in a certain sense the problem to find the square roots of a symmetric 2D matrix A.

We first define a symmetric 2D matrix A by means of an arbitrary 2D matrix U as follows:

$$A \equiv UU^{\mathsf{T}} = \begin{pmatrix} U_{xx}^2 + U_{xy}^2 & U_{xx}U_{yx} + U_{xy}U_{yy} \\ U_{xx}U_{yx} + U_{xy}U_{yy} & U_{yx}^2 + U_{yy}^2 \end{pmatrix} = \begin{pmatrix} A_{xx} & A_{xy} \\ A_{xy} & A_{yy} \end{pmatrix}$$
$$A = A^{\mathsf{T}} \qquad A_{xx}A_{yy} - A_{xy}^2 = |A| = |U||U^{\mathsf{T}}| = |U|^2 = (U_{xx}U_{yy} - U_{xy}U_{yx})^2. \tag{9.1}$$

For a given arbitrary matrix U, the symmetric matrix A is uniquely determined. If U is unimodular then A is also unimodular. However, for given symmetric matrix $A = A^{\mathsf{T}}$, a matrix U defined by $UU^{\mathsf{T}} = A$ is not uniquely determined. The determination of U from A is the determination of the square roots of A in the sense that a quadratic form $\bar{s}A\bar{s} = \bar{s}UU^{\mathsf{T}}\bar{s}$ is represented as the scalar product of a row vector $\bar{s}U$ with its corresponding column vector $U^{\mathsf{T}}\bar{s}$ for arbitrary vectors $\bar{s}$. The considered 'square root problem' is not identical to another possible 'square root problem' which is of the form $U^2 = A$. We, however, need the solution of the 'square root problem' $UU^{\mathsf{T}} = A$. This solution contains a free parameter denoted by λ and can be written according to (many other parametrizations are possible)

$$U = \begin{pmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{pmatrix} = \frac{1}{\sqrt{\lambda^2 A_{xx} + 2\lambda\sqrt{|A|} + A_{yy}}} \begin{pmatrix} \lambda A_{xx} + \sqrt{|A|} & A_{xy} \\ \lambda A_{xy} & A_{yy} + \lambda\sqrt{|A|} \end{pmatrix} \tag{9.2}$$

where the sign of $\sqrt{|A|}$ can be chosen arbitrarily but it has to be the same in all parts of this solution. One can check by insertion into (9.1) that this is a solution but its straightforward derivation from (9.1) showing in addition that it is the general solution is more difficult and we do not write it down. Whereas the matrix $A = A^{\mathsf{T}}$ contains, in general, three complex components, an arbitrary matrix U contains, in general, four complex components and this makes it understandable that there appears a free parameter λ when determining U from $UU^{\mathsf{T}} = A$. By choosing $\lambda = 1$, one obtains from (9.2) a symmetric matrix $U = U^{\mathsf{T}}$ for U and by choosing $\lambda = 0$ or $\lambda = \infty$ a right triangular or a left triangular matrix for U.

If A possesses diagonal form $A_{xx} = a$, $A_{yy} = b$, $A_{xy} = 0$, then by the substitutions $A_{xy} \equiv \varepsilon$ and $\lambda \equiv -\sqrt{b/a} + (\mu\varepsilon)/(\sqrt{1-\mu^2}\,a)$, one finds from (9.2) in the limiting case $\varepsilon \to 0$

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \Leftrightarrow \quad U = \begin{pmatrix} \mu\sqrt{a} & \sqrt{1-\mu^2}\sqrt{a} \\ -\sqrt{1-\mu^2}\sqrt{b} & \mu\sqrt{b} \end{pmatrix} \tag{9.3}$$

with an arbitrary parameter μ. For $\mu = 1$, one obtains from (9.3) a diagonal matrix U. Another special case is the following correspondence for square roots of the spin matrix σ_1:

$$A = c\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i}c\sigma_1 \quad \Leftrightarrow \quad U = \frac{1}{\sqrt{2\lambda\sqrt{c^2}}} \begin{pmatrix} \sqrt{c^2} & \mathrm{i}c \\ \mathrm{i}\lambda c & \lambda\sqrt{c^2} \end{pmatrix} \tag{9.4}$$

where $\sqrt{c^2}$ means that both signs $\pm c$ are admissible if it is only the same sign in the different parts of the expression. For $c = 1$ and by choosing $\lambda = -\mathrm{i}$, one obtains $U = Z$ (see (7.1)).

10. Relations of Hermite 2D polynomials to two-variable Hermite polynomials

We distinguished the Hermite 2D polynomials introduced in section 3 from the usual two-variable Hermite polynomials by name and notation since they are not identical to each other. We now establish the corresponding relations. This can be done most easily by comparing the generating functions.

There are two kinds of two-variable Hermite polynomials $H_{m,n}^{R}(x, y)$ and $G_{m,n}^{R}(x, y)$ which are usually defined with the help of a symmetric 2×2-matrix R by the following generating function for the first kind (e.g. [11, 18, 22]):

$$\exp\left(\bar{s}R\bar{x}' - \tfrac{1}{2}\bar{s}R\bar{s}\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{s^m t^n}{m!n!} H_{m,n}^{R}(x', y') \qquad R = R^{\mathsf{T}} \tag{10.1}$$

and by the following generating function for the second kind [11]:

$$\exp\left(\bar{s}\bar{x}' - \tfrac{1}{2}\bar{s}R^{-1}\bar{s}\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{s^m t^n}{m!n!} G_{m,n}^{R}(x', y') \qquad R = R^{\mathsf{T}}. \tag{10.2}$$

The vector abbreviations are the same as in (8.3) but here we have primed the independent variables because they cannot be identified with the independent variables in the Hermite 2D polynomials without transformations. In most cases, only the two-variable Hermite polynomials $H_{m,n}^{R}(x, y)$ of the first kind are used. The second kind is defined for the purpose of the orthonormalization of the first kind in the form of biorthogonality relations. Many special cases and representations of $H_{m,n}^{R}(x, y)$ are considered in [13–18, 22].

We now compare these generating functions with the generating function (8.4) for the Hermite 2D polynomials and find from (10.1) the correspondences

$$R\bar{x}' = 2U\bar{x} \qquad R = 2UU^{\mathsf{T}}. \tag{10.3}$$

From this we find the following relations:

$$\begin{aligned} H_{m,n}(U; x, y) &= H_{m,n}^{2UU^{\mathsf{T}}}\left(\frac{xU_{yy} - yU_{yx}}{|U|}, \frac{-xU_{xy} + yU_{yy}}{|U|}\right) \\ H_{m,n}^{R}(x, y) &= H_{m,n}(U; xU_{xx} + yU_{yx}, xU_{xy} + yU_{yy}) \qquad R = 2UU^{\mathsf{T}}. \end{aligned} \tag{10.4}$$

This means the following. The Hermite 2D polynomial $H_{m,n}(U; x, y)$ can be easily, but not simply, expressed for a given matrix U by a two-variable Hermite polynomial $H_{m,n}^{R}(x', y')$. However, to express $H_{m,n}^{R}(x, y)$ for given matrix R by a certain $H_{m,n}(U; x', y')$ requires the solution of the 'square root problem' $UU^{\mathsf{T}} = R/2 \equiv A$ for the determination of a possible matrix U as dealt with in the preceding section. Therefore, starting from the two-variable Hermite polynomials $H_{m,n}^{R}(x, y)$, it is a highly non-trivial problem to change the definition in such a way that it provides a set of orthonormalized polynomials similarly to (5.9).

From (10.2) together with (8.4), we find the correspondences

$$\bar{x}' = 2U\bar{x} \qquad R^{-1} = 2UU^{\mathsf{T}}. \tag{10.5}$$

This leads to

$$\begin{aligned} H_{m,n}(U; x, y) &= G_{m,n}^{\frac{1}{2}(UU^{\mathsf{T}})^{-1}}\left(2(U_{xx}x + U_{xy}y), 2(U_{yx}x + U_{yy}y)\right) \\ G_{m,n}^{R}(x, y) &= H_{m,n}\left(U; \frac{U_{yy}x - U_{xy}y}{2|U|}, \frac{-U_{yx}x + U_{xx}y}{2|U|}\right) \qquad R^{-1} = 2UU^{\mathsf{T}}. \end{aligned} \tag{10.6}$$

Here we have to solve a 'square root problem' of the form $UU^{\mathsf{T}} = R^{-1}/2 \equiv A$ for the determination of U.

The main advantage of the Hermite 2D polynomials in comparison to the two-variable Hermite polynomials is that they obey orthonormality and completeness relations, whereas the two-variable Hermite polynomials need a second kind of polynomials for the formulation of biorthonormality relations. A further advantage of the definition of the Hermite 2D polynomials is that apart from the degenerate case of the vanishing determinant of U it shows that a general matrix U does not lead essentially to other polynomials than the corresponding unimodular matrices $U' = U/\sqrt{|U|}$ but only to their multiplication with powers of $\sqrt{|U|}$. This is contained in the two-variable Hermite polynomials in a very entangled form.

11. Conclusion

With the present paper together with the preceding two papers [2, 3], we have realized in a first approximation our programme connected with the introduction of a general set of Hermite and Laguerre 2D polynomials and corresponding 2D functions for the degenerate 2D harmonic oscillator. This programme can be characterized by the following scheme (generalization from the left and from the right to the centre of the lines; transition from sets of polynomials to orthonormalized and complete sets of functions from above to below; representation by real variables on the left-hand side and by complex variables on the right-hand side):

$$\begin{array}{ccccccc} H_m(x)H_n(y) & \leftarrow & H_{m,n}(U;x,y) & \Leftrightarrow & L_{m,n}(U;z,z^*) & \rightarrow & L_{m,n}(z,z^*) \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ h_m(x)h_n(y) & \leftarrow & h_{m,n}(U;x,y) & \Leftrightarrow & l_{m,n}(U;z,z^*) & \rightarrow & l_{m,n}(z,z^*). \end{array} \tag{11.1}$$

The special Hermite and Laguerre 2D polynomials and functions are obtained from the general ones by setting $U = I$, where I is the 2D identity matrix which is then omitted in the notation. The Hermite 2D polynomials $H_{m,n}(U;x,y)$ are not identical to the usual two-variable Hermite polynomials and the corresponding relations to each other are discussed. Our Hermite 2D functions (and Laguerre 2D functions too) form an orthonormalized and complete set of 2D functions, whereas one has two different kinds of usual two-variable Hermite functions which are biorthogonal in a dual way. Roughly speaking, our set of Hermite 2D polynomials is obtained by using the square root of the matrix which figures in the usual two-variable Hermite polynomials. Our programme can be briefly characterized as the $SL(2,C)$ unification of Hermite and Laguerre 2D polynomials. Most important for applications are sublevels of transformations, for example, the sublevel of the $SU(2)$ unification and sometimes the sublevel of the $SU(1,1)$ unification of the considered polynomials which can be obtained by corresponding specialization of the 2D matrices U.

Due to the rich material, we could not consider here all the problems which seem to be interesting for the introduced Hermite 2D and Laguerre 2D polynomials and functions and restricted ourselves to the most important ones from our point of view. These were different explicit representations, the orthonormality relations, the simplest generating functions and the relations to the usual two-variable Hermite polynomials. Other problems of interest which have not yet been considered are, for example, the introduction of annihilation and creation operators and recursion relations [35], transformation relations involving the arguments of the polynomials, specializations of the matrix U leading to simplifications of the polynomials and Fourier and Radon transformations of the Hermite and Laguerre 2D functions. The group $SL(2,C) \sim Sp(2,C)$ contains three complex parameters and a parametrization by complex 3-vectors becomes possible which is not considered here. There are also categories of one- and two-dimensional integrals with parameters and integral transformations which lead to Hermite and Laguerre 2D polynomials. They can be taken as integral representations of these polynomials on one hand and as the evaluation of interesting integrals on the other hand. Last but not least, we expect interesting applications such as, for example, in quantum optics to the lossless beamsplitter and to general two-mode polarization of light and, furthermore, to the representation of ordered moments for displaced squeezed thermal states, where many interesting results already exist in the literature in different forms. The generalization of the Hermite 2D polynomials and functions in the representation by real variables (x, y) to Hermite νD polynomials and functions is possible in a way which is briefly indicated at the end of section 8. These are some of the problems arising in connection with the new definition of Hermite and Laguerre 2D polynomials. Having now outlined the contours of the new concept of the introduction of Hermite and Laguerre 2D polynomials, the author invites the reader to take part in the discussion of the many interesting properties of these polynomials and to consider applications. Much remains to be done.

Acknowledgments

The author would like to express his gratitude to colleagues who contributed by discussions or by correspondence concerning the preceding papers to the improvement of the present paper. These are, in particular, A Vourdas from Liverpool, V V Dodonov and V I Man'ko from Moscow, Hong-yi Fan from Shanghai, E Abramochkin and V Volostnikov from Samara, M M Nieto from Los Alamos for sending his papers and G Le Caer from Nancy.

Appendix A. A group-theoretical addition theorem for Jacobi polynomials

In this appendix, we derive a new relation for the Jacobi polynomials which results from the composition of two non-degenerate transformations U and V of the form (2.7) to a new transformation $W = UV$. The set of these 2D linear transformations forms the group $GL(2, C)$. Due to (2.11), we can restrict ourselves without loss of generality to unimodular transformations U' and V' and therefore also to $W' = U'V'$ and consider the transformations

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} W_{11}' & W_{12}' \\ W_{21}' & W_{22}' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} U_{11}' & U_{12}' \\ U_{21}' & U_{22}' \end{pmatrix} \begin{pmatrix} V_{11}' & V_{12}' \\ V_{21}' & V_{22}' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$W' = U'V' \qquad |U'| = 1 \qquad |V'| = 1 \quad \Rightarrow \quad |W'| = 1. \tag{A.1}$$

The set of these transformations forms the group $SL(2, C) \sim Sp(2, C)$ of 2D complex unimodular or symplectic transformations. If we treat the transformation of the powers of (x_1', x_2') by using the product of matrices U' and V' then we obtain from (2.13)

$$x_1'^m x_2'^n = \sum_{j=0}^{m+n} U_{12}'^{m-j} U_{22}'^{n-j} P_j^{(m-j,n-j)} \left(1 + 2U_{12}'U_{21}'\right)$$
$$\times \sum_{k=0}^{m+n} V_{12}'^{j-k} V_{22}'^{m+n-j-k} P_k^{(j-k,m+n-j-k)} \left(1 + 2V_{12}'V_{21}'\right) x_1^k x_2^{m+n-k}. \tag{A.2}$$

On the other hand, with the product matrix $W' = U'V'$ of the transformations, it follows

$$x_1'^m x_2'^n = \sum_{k=0}^{m+n} W_{12}'^{m-k} W_{22}'^{n-k} P_k^{(m-k,n-k)} \left(1 + 2W_{12}'W_{21}'\right) x_1^k x_2^{m+n-k}. \tag{A.3}$$

By comparison of the expressions in front of $x_1^k x_2^{m+n-k}$ for each k in (A.2) and (A.3), one obtains the identities

$$W_{12}'^{m-k} W_{22}'^{n-k} P_k^{(m-k,n-k)} \left(1 + 2W_{12}'W_{21}'\right) = \sum_{j=0}^{m+n} U_{12}'^{m-j} U_{22}'^{n-j} V_{12}'^{j-k} V_{22}'^{m+n-j-k}$$
$$\times P_j^{(m-j,n-j)} \left(1 + 2U_{12}'U_{21}'\right) P_k^{(j-k,m+n-j-k)} \left(1 + 2V_{12}'V_{21}'\right)$$
$$= \frac{1}{k!(m+n-k)!} \sum_{j=0}^{m+n} j!(m+n-j)! U_{12}'^{m-j} U_{22}'^{n-j} V_{21}'^{k-j} V_{22}'^{m+n-k-j}$$
$$\times P_j^{(m-j,n-j)} \left(1 + 2U_{12}'U_{21}'\right) P_j^{(k-j,m+n-k-j)} \left(1 + 2V_{12}'V_{21}'\right) \tag{A.4}$$

where the components of W' expressed by the components of U' and V' can be taken from (A.1). The second form can be obtained by using the first of the transformation relations for the Jacobi polynomials given in (2.4). In this last form, both Jacobi polynomials have the same lower index j which is the summation index. Relation (A.4) is a kind of addition theorem for the Jacobi polynomials. It connects Jacobi polynomials with different arguments which are determined by a matrix product and it contains three discrete parameters (m, n) and $k = 0, \ldots, m+n$.

We consider the special case $W' = U'V' = I$ in (A.4). In this case, one has

$$V' = U'^{-1} = \begin{pmatrix} U'_{22} & -U'_{12} \\ -U'_{21} & U'_{11} \end{pmatrix} \qquad |V'| = |U'|^{-1} = 1 \tag{A.5}$$

since U' is unimodular. Due to

$$P_k^{(m-k,n-k)}(1) = \frac{m!}{k!(m-k)!} \qquad \lim_{W\to I} W_{12}^{m-k} = \delta_{m-k,0} \qquad m \geqslant k \tag{A.6}$$

following from (2.3) and from the assumption $W' = I$, one finds from (A.5) the identity

$$\begin{aligned} \delta_{m-k,0} = {} & \sum_{j=0}^{m+n} (U'_{11}U'_{22})^{n-j}(-1)^{j-k} P_j^{(m-j,n-j)}(1+2U'_{12}U'_{21}) \\ & \times P_k^{(j-k,m+n-j-k)}(1+2U'_{12}U'_{21}) \\ = {} & \frac{1}{m!n!}\sum_{j=0}^{m+n} j!(m+n-j)!\,(-U'_{12}U'_{21})^{m-j}(U'_{11}U'_{22})^{n-j} \\ & \times P_j^{(m-j,n-j)}\left(1+2U'_{12}U'_{21}\right) P_j^{(k-j,m+n-k-j)}\left(1+2U'_{12}U'_{21}\right) \qquad m \geqslant k. \end{aligned} \tag{A.7}$$

This means that in the special case $V' = U'^{-1}$, one obtains from (A.4) a summation relation over products of Jacobi polynomials with equal arguments. In particular, for $k = m$ and by using $U'_{11}U'_{22} - U'_{12}U'_{21} = 1$, one obtains from the second part of this relation

$$\frac{1}{m!n!}\sum_{j=0}^{m+n} j!(m+n-j)!(-U'_{12}U'_{21})^{m-j}(1+U'_{12}U'_{21})^{n-j}\left(P_j^{(m-j,n-j)}(1+2U'_{12}U'_{21})\right)^2 = 1. \tag{A.8}$$

It contains two free integer parameters (m, n) and one free continuous parameter $U'_{12}U'_{21}$ (or $U'_{11}U'_{22}$). I have checked this identity for different parameters by computer and did not find any contradictions.

It is easy to generalize the relations of this appendix to arbitrary 2D matrices (U, V, $W = UV$) but it is inconvenient to write this down.

Appendix B. Proof of the orthonormalization of the special Laguerre 2D functions

An indirect proof of the orthonormalization (5.14) of the special Laguerre 2D functions relying on known orthogonality relations for the usual (1D) Laguerre polynomials was briefly described in [2] and a possible derivation from the obtained differential equations was announced. We will give here a more direct proof which reveals some interesting relations for Jacobi polynomials. This proof uses the following two identities:

$$\frac{1}{\pi}\int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}z^* \exp(-zz^*) z^k z^{*l} = k!\delta_{k,l} \qquad (k, l = 0, 1, \ldots) \tag{B.1}$$

with an integral which is easily evaluated in polar coordinates and

$$\sum_{j=0}^{\{m,n\}} \frac{(-1)^j (k+m+n-j)!}{j!(m-j)!(n-j)!} = \frac{(k+m)!(k+n)!}{k!m!n!} \equiv \frac{(k+n)!}{n!} P_m^{(k,n-m)}(1). \tag{B.2}$$

The identity in (B.2) is interesting for itself because it is related to a representation of the Jacobi polynomials which happens to appear in applications but is little known. However, before showing this let us first finish the proof of the orthonormalization (5.14) by using (B.1) and (B.2) and, furthermore, the binomial formula.

Starting from the definition of the Laguerre 2D functions, one finds

$$\begin{aligned}
\int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}z^* \left(l_{k,l}(z,z^*)\right)^* l_{m,n}(z,z^*) &= \frac{1}{\pi\sqrt{k!l!m!n!}} \int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}z^* \exp(-zz^*) \\
&\quad \times \sum_{i=0}^{\{k,l\}} \sum_{j=0}^{\{m,n\}} \frac{k!l!m!n!(-1)^{i+j}}{i!(k-i)!(l-i)!j!(m-j)!(n-j)!} z^{l+m-i-j} z^{*\,k+n-i-j} \\
&= \delta_{l+m,k+n}\sqrt{k!l!m!n!} \sum_{i=0}^{\{k,l\}} \frac{(-1)^i}{i!(k-i)!(l-i)!} \sum_{j=0}^{\{m,n\}} \frac{(-1)^j (l+m-i-j)!}{j!(m-j)(n-j)!} \\
&= \delta_{l+m,k+n}\sqrt{\frac{k!l!}{m!n!}} \frac{1}{(l-n)!} \sum_{i=0}^{l-n} \frac{(-1)^i (l-n)!}{i!(l-n-i)!} \\
&= \delta_{k,m}\delta_{l,n}.
\end{aligned} \tag{B.3}$$

Thus the orthonormalization of the special Laguerre 2D functions is proved.

We now decompose $\{(u-1)/2\}^{j-k} = \{(u+1)/2 - 1\}^{j-k}$ in the definition (2.2) of the Jacobi polynomials and apply the binomial formula to obtain a representation by powers of $(u+1)/2$ and then we change the order of summations in the arising double sum. This yields

$$P_j^{(\alpha,\beta)}(u) = (j+\beta)! \sum_{l=0}^{j} \frac{(-1)^l}{l!(j-l)!} \left(\tfrac{1}{2}(u+1)\right)^{j-l} \sum_{k=0}^{j-l} \frac{(j-l)!(j+\alpha)!}{k!(j-l-k)!(j+\alpha-k)!(\beta+k)!}. \tag{B.4}$$

The inner sum can be evaluated by means of the convolution formula for the binomial coefficients following from the product of two binomials:

$$\sum_{k=0}^{\{m,n\}} \frac{m!n!}{k!(m-k)!(n-k)!(l+k)!} = \frac{(l+m+n)!}{(l+m)!(l+n)!} \equiv \frac{n!}{(n+l)!} \lim_{|u|\to\infty} \left(\frac{2}{u}\right)^n P_n^{(m-n,l)}(u). \tag{B.5}$$

By using this, one obtains from (B.4) the following representation of the Jacobi polynomials (the only place we found a similar relation is [26] (22.3.2), author, Hochstrasser)

$$\begin{aligned}
P_j^{(\alpha,\beta)}(u) &= \frac{(j+\beta)!}{(j+\alpha+\beta)!} \sum_{l=0}^{j} \frac{(-1)^l (2j+\alpha+\beta-l)!}{l!(j-l)!(j+\beta-l)!} \left(\tfrac{1}{2}(u+1)\right)^{j-l} \\
&= (-1)^j P_j^{(\beta,\alpha)}(-u).
\end{aligned} \tag{B.6}$$

Now, by setting $u = 1$ in (B.6) and by the substitutions $j \to m, \alpha \to k, \beta \to n-m$ and by using (2.3) for the Jacobi polynomials with argument $u = 1$, one obtains the identity (B.2).

From the two very different representations of the Jacobi polynomials in (2.2) and (B.6), one obtains the following relation in the limiting case $|u| \to \infty$:

$$\lim_{|u|\to\infty} \left(\frac{2}{u}\right)^j P_j^{(\alpha,\beta)}(u) = \sum_{k=0}^{j} \frac{(j+\alpha)!(j+\beta)!}{k!(j+\alpha-k)!(j-k)!(\beta+k)!} = \frac{(2j+\alpha+\beta)!}{j!(j+\alpha+\beta)!}. \tag{B.7}$$

For $\alpha \to m-j, \beta \to n-j$, this yields binomial coefficients in the form

$$\lim_{|u|\to\infty} \left(\frac{2}{u}\right)^j P_j^{(m-j,n-j)}(u) = \frac{(m+n)!}{j!(m+n-j)!}. \tag{B.8}$$

Furthermore, it is interesting that the identity (B.7) becomes equal to the summation formula (B.5) after the substitutions $j \to n, \alpha \to m-n, \beta \to l$ and subsequent multiplication by $n!/(n+l)!$. Thus both summation formulae (B.2) and (B.5) considered in this appendix possess a relation to the Jacobi polynomials which is revealed by taking into account the different representations (2.2) and (B.6) of these polynomials.

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